On the homotopy classification of elliptic operators on stratified manifolds*

V.E. Nazaikinskii, A.Yu. Savin, and B.Yu. Sternin

August 14, 2006

Abstract

We find the stable homotopy classification of elliptic operators on stratified manifolds. Namely, we establish an isomorphism of the set of elliptic operators modulo stable homotopy and the K-homology group of the singular manifold. As a corollary, we obtain an explicit formula for the obstruction of Atiyah–Bott type to making interior elliptic operators Fredholm.

Contents

*Supported in part by RFBR grants 05-01-00982 and 06-01-00098, by President of the Russian Federation grant MK-1713.2005.1, and by the DFG project 436 RUS 113/849/0-1® "K-theory and Noncommutative Geometry of Stratified Manifolds."

1 Introduction

In the classical paper [1], Atiyah observed that the elements of K-homology groups of a compact space X are realized as *abstract elliptic operators* on X. (Recall that the latter are Fredholm operators acting in $C(X)$-modules and commuting with multiplications by functions modulo compact operators.) Moreover, Kasparov [2] and Brown–Douglas–Fillmore [3] showed that one can obtain not only elements of K-homology groups but also K-homology as a generalized homology theory if one factorizes abstract elliptic operators by a special equivalence relation known as stable homotopy.

However, it turns out that, at least for sufficiently smooth spaces, the K-homology group can already be obtained if, instead of abstract elliptic operators, we restrict ourselves to *differential* or *pseudodifferential* operators, which are natural in the theory of partial differential equations. For example, on closed smooth manifolds it suffices to consider pseudodifferential operators (ψDO in what follows), and if the manifold is equipped with a *spin*^{c}-structure, then one can take the class of (twisted) Dirac operators. This example serves as a motivation for the natural problem of comparing the K-homology group with the group generated by elliptic operators on singular manifolds (cf. Singer's problem in [4]).

The main theorem of the present paper establishes a group isomorphism

$$\operatorname{Ell}(X) \simeq K_0(X) \tag{1}$$

on any compact stratified manifold X with arbitrary number of strata, where $\operatorname{Ell}(X)$ is the group generated by elliptic pseudodifferential operators on X modulo stable homotopy. Special cases of such isomorphisms were obtained in [5–9] for manifolds with two strata.

The isomorphism (1) enables one to apply topological methods of K-homology theory in elliptic theory. As examples of such applications, in Sec. 8 we compute the obstruction of Atiyah–Bott type to making interior elliptic operators Fredholm and obtain a generalization of the cobordism invariance of the index.

Apart from already mentioned applications to elliptic operators, the isomorphism (1) has an interesting interpretation in the framework of noncommutative geometry. More precisely, the groups in (1) can be represented as the left- and right-hand sides in the Baum–Connes conjecture [10]. The point is that from the viewpoint of noncommutative geometry the algebra of ψDO on a stratified manifold is related to a certain groupoid (see [11,12]). Moreover, the group $\operatorname{Ell}(X)$ is related to the K-group of the groupoid C^*-algebra [13]. The Baum–Connes conjecture claims that the latter K-group is isomorphic to the topological K-group of the classifying space of the groupoid (see [14]). Explicit computations for simplest stratified manifolds show that the K-group of the classifying space is isomorphic to $K_0(X)$, that is, the right-hand side of (1). It would be of interest to make further comparison of (1) with the Baum–Connes map.

We are grateful to Prof. T. Fack (Lion) for pointing out the relation with the Baum–Connes isomorphism for groupoids. The results of this paper were presented at the International conference "Workshop on index theory" in Münster (November 4–8, 2005).

2 Stratified manifolds and ψDO

In this section, we define a class of manifolds and a class of pseudodifferential operators on them and study ellipticity and Fredholm property for such operators. These topics are well known in the literature (e.g., see [15–17]). Hence the exposition will be rather concise.

2.1 Stratified manifolds

In the sequel, we adopt the following terminology. A *manifold with singularities* is a triple

$$\pi : M \longrightarrow \mathcal{M},$$

where $\mathcal{M}$ is a Hausdorff space, M is a manifold with corners,[1] and π is a continuous projection. The manifold M is called the *blow-up* of $\mathcal{M}$. We shall not discuss the uniqueness of the blow-up and always assume when speaking about manifolds with singularities that the triple $(\pi : M \longrightarrow \mathcal{M})$ is fixed.

A *diffeomorphism* of such manifolds is a pair $f : \mathcal{M}_1 \to \mathcal{M}_2$, $\widetilde{f} : M_1 \to M_2$ such that f is a homeomorphism, $\widetilde{f}$ is a diffeomorphism, and the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{\widetilde{f}} & M_2 \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ \mathcal{M}_1 & \xrightarrow[f]{} & \mathcal{M}_2, \end{array}$$

where $\pi_{1,2}$ are the natural projections, commutes. A special class of manifolds with singularities can be obtained as follows: given a smooth Riemannian metric on a manifold M with corners, nondegenerate in the interior and possibly degenerate at the boundary, define $\mathcal{M}$ as the quotient of M by the following equivalence relation determined by the metric: two points are equivalent if the distance between them in our metric is zero. (Of course, the Hausdorff property of $\mathcal{M}$ should be additionally assumed.)

Now let us describe the class of manifolds with singularities to be studied in the present paper, namely, stratified manifolds. The definition is by induction.

Definition 2.1. A *filtration of length* k on a topological space $\mathcal{M}$ is a sequence

$$\mathcal{M} = \mathcal{M}_k \supset \mathcal{M}_{k-1} \supset \ldots \supset \mathcal{M}_0 \tag{2}$$

of embedded subspaces, closed in $\mathcal{M}$, such that each $\mathcal{M}_j$ is contained in the closure of $\mathcal{M}^\circ_{j+1} = \mathcal{M}_{j+1} \setminus \mathcal{M}_j$, $j = 0, \ldots, k-1$.

Definition 2.2 (base of induction). *A stratified manifold of length* 0 is an arbitrary smooth manifold.

In this case, we set $M = \mathcal{M}$ and $\pi = \mathrm{id}$, and the blow-up M is a smooth manifold without boundary.

[1] Recall that a manifold of dimension n with corners is a Hausdorff space locally homeomorphic to the product $\overline{\mathbb{R}}^k_+ \times \mathbb{R}^{n-k}$, $0 \le k \le n$.

Definition 2.3 (inductive step). A *stratified manifold of length* $k > 0$ is a Hausdorff space $\mathcal{M}$ equipped with a filtration (2) such that

1. $\mathcal{M}_0$ is a smooth manifold.

2. $\mathcal{M} \setminus \mathcal{M}_0$ is equipped with the structure of a stratified manifold of length $k-1$ with respect to the filtration

$$\mathcal{M} \setminus \mathcal{M}_0 = \mathcal{M}_r \setminus \mathcal{M}_0 \supset \mathcal{M}_{k-1} \setminus \mathcal{M}_0 \supset \ldots \supset \mathcal{M}_1 \setminus \mathcal{M}_0.$$

3. Over $\mathcal{M}_0$ there is a given bundle with fiber K_Ω, where K_Ω is a cone with base Ω that is a compact stratified manifold of length less than or equal to $k-1$. There is also a given homeomorphism of a neighborhood $U \subset \mathcal{M}$ of $\mathcal{M}_0$ onto a neighborhood of the subbundle formed by the cone tips; this homeomorphism restricts to the identity map on $\mathcal{M}_0$.

4. The structure described in Condition 3 should be compatible with that in Condition 2 on $\mathcal{M} \setminus \mathcal{M}_0$ in the sense described below.

First, note that one can prove by induction that the definition implies that

- The sets $\mathcal{M}_j \setminus \mathcal{M}_{j-1} \simeq M_j^\circ$ (called open strata) are smooth manifolds for all $j = 0, \ldots, k$.[2]

- Given any open stratum M_j°, $j < k$, there exists a neighborhood $U_j \subset \mathcal{M}$ homeomorphic to a bundle with fiber K_{Λ_j}, where the base of the cone Λ_j is a stratified manifold of length $\leq k - j - 1$.

Let us give the precise statement of Conditions 3 and 4.
The cone K_Ω in Condition 3 is the set

$$K_\Omega = \{\overline{\mathbb{R}}_+ \times \Omega\} \;/\; \{\{0\} \times \Omega\}.$$

We require that the transition functions of the bundle with fiber K_Ω leave the variable $r \in \overline{\mathbb{R}}_+$ invariant and are induced by *diffeomorphisms of the stratified manifold* Ω of length $\leq k-1$, i.e., diffeomorphisms of manifolds with singularities preserving the stratification and the fibrations of neighborhoods of strata into cones. Hence we actually require that our bundle is obtained from a bundle over $\mathcal{M}_0$ with fiber Ω by fiberwise conification.

We claim that the complement $U \setminus \mathcal{M}_0$ is equipped with a natural structure of a stratified manifold of length $\leq k-1$ as an open subset of the bundle over $\mathcal{M}_0$ with fiber

$$K_\Omega^\circ \simeq \mathbb{R}_+ \times \Omega$$

equal to the cone minus the cone tip. Indeed, the local trivializations of this bundle are $V \times \mathbb{R}_+ \times \Omega$, where $V \subset \mathbb{R}^l$, $l = \dim \mathcal{M}_0$ is a coordinate patch on $\mathcal{M}_0$. Therefore, they are stratified manifolds of the same length as Ω, i.e., $\leq k-1$. Now we can take

[2] We set $\mathcal{M}_{-1} = \emptyset$ for convenience. These sets are the interior parts of the corresponding blow-ups, which are smooth manifolds with corners.

$V \times \mathbb{R}_+ \times \Omega_j$ for strata in $V \times \mathbb{R}_+ \times \Omega$, where Ω_j are the corresponding strata in Ω, and bundles with conical fibers of neighborhoods of these strata are obtained from the bundles of neighborhoods of the corresponding strata in Ω by taking products with $V \times \mathbb{R}_+$.

Let us clarify Condition 4. By virtue of the preceding, $U \setminus \mathcal{M}_0$ is equipped with two stratified manifold structures: the first is the restriction of the corresponding structure to $\mathcal{M} \setminus \mathcal{M}_0$, and the second comes from the bundle. *The compatibility condition requires that these two structures coincide* (i.e., the identity map is a diffeomorphism).

Finally, let us define the blow-up of $\mathcal{M}$. Let $\widetilde{\pi} : \widetilde{M} \longrightarrow \mathcal{M} \setminus \mathcal{M}_0$ be the projection of the blow-up $\mathcal{M} \setminus \mathcal{M}_0$ (it is defined by the inductive hypothesis). The blow-up M of $\mathcal{M}$ is obtained if we augment $\widetilde{M}$ with a certain set "at infinity over $\mathcal{M}_0$." Hence to describe M and the projection $\pi : M \longrightarrow \mathcal{M}$ it suffices to study what happens over U. We can assume that U itself is fibered over $\mathcal{M}_0$; then $\widetilde{\pi}^{-1}(U \setminus \mathcal{M}_0)$ is fibered over $\mathcal{M}_0$ by the composition map

$$\widetilde{\pi}^{-1}(U \setminus \mathcal{M}_0) \xrightarrow{\widetilde{\pi}} U \setminus \mathcal{M}_0 \longrightarrow \mathcal{M}_0.$$

The structure of this bundle is easy to describe in local trivializations; the bundle has the form

$$V \times \mathbb{R}_+ \times \widetilde{\Omega} \xrightarrow{\mathrm{id} \times \mathrm{id} \times p} V \times \mathbb{R}_+ \times \Omega \longrightarrow V,$$

where $p : \widetilde{\Omega} \longrightarrow \Omega$ is the projection of the blow-up of the cone base. In this local trivialization, the blow-up M of $\mathcal{M}$ is simply defined by adding the point $r = 0$ to the second factor, i.e., by passing from $\mathbb{R}_+$ to $\overline{\mathbb{R}}_+$; the projection takes each point $(v, 0, \omega)$ to v.

Now the meaning of Definition 2.3 is completely clear.

Each of the strata $\mathcal{M}_j$ is a stratified manifold itself (of length j). It has a blow-up M_j and the corresponding projection $p_j : M_j \longrightarrow \mathcal{M}_j$.

Remark 2.1. By Condition 3, conical bundles are defined over the open strata M_j° in $\mathcal{M}$. However, Condition 4 readily implies that these bundles can be canonically extended to the closed strata M_j. This remark will be important below, since the operator-valued symbols of our pseudodifferential operators will be defined on the closed strata.

Metrics, measures, and L^2-spaces. We introduce some natural metrics and measures on stratified manifolds. They are used in the definition of spaces of square integrable functions.

First, we give an inductive definition of metrics. On a stratified manifold of length 0 we consider an arbitrary Riemannian metric. Let us describe the inductive step. Given a metric $d\widetilde{\omega}^2$ on $\widetilde{\Omega}$ supplied by the induction hypothesis, we can define a metric locally on $V \times \mathbb{R}_+ \times \widetilde{\Omega}$ by the formula

$$ds^2 = dv^2 + dr^2 + r^2 d\widetilde{\omega}^2. \tag{3}$$

Globally on M, the metric is obtained with the use of a partition of unity from the local expressions described above in a neighborhood of $\mathcal{M}_0$ and the metric $\widetilde{d\rho}^2$ defined on $\mathcal{M} \setminus \mathcal{M}_0$ by the induction hypothesis outside a slightly smaller neighborhood of $\mathcal{M}_0$. Metrics of this form are called *edge-degenerate*.

The metric naturally produces the measure defined as the volume element equal to unity on an orthonormal frame. In terms of the inductive formula (3), the corresponding formula for the volume form is

$$d\,\mathrm{vol} = r^n dv\, dr\, d\,\mathrm{vol}_\Omega,$$

where $d\,\mathrm{vol}_\Omega$ is the volume form on Ω (known by the induction hypothesis) and $n = \dim \Omega$ is the dimension of Ω.

From now on, all operators on $\mathcal{M}$ are considered in the space

$$L^2(\mathcal{M}) \equiv L^2(\mathcal{M}, d\,\mathrm{vol}),$$

and the operators on the cone K_Ω are considered in the space

$$L^2(K_\Omega) \equiv L^2(K_\Omega, r^n\, dr\, d\,\mathrm{vol}_\Omega).$$

The cotangent bundle. We define a space $\mathrm{Vect}_\mathcal{M}$ of vector fields on $\mathcal{M}$. If $\mathcal{M}$ is smooth, then $\mathrm{Vect}_\mathcal{M}$ is the space of all vector fields on $\mathcal{M}$. In the general case, $\mathrm{Vect}_\mathcal{M}$ is defined as a $C^\infty(M)$-module locally as follows. Over the product $V \times \mathbb{R}_+ \times \widetilde{\Omega}$, the space $\mathrm{Vect}_\mathcal{M}$ is formed by vector fields of the form

$$\theta = a\frac{\partial}{\partial v} + b\frac{\partial}{\partial r} + \frac{1}{r}\theta_1,$$

where a and b are smooth functions and $\theta_1 \in \mathrm{Vect}_\Omega$.

The metric ds^2 defines a $C^\infty(M)$-valued pairing on $\mathrm{Vect}_\mathcal{M}$, and the formula

$$\langle \varphi(\theta), \mu \rangle = ds^2(\theta, \mu)$$

gives a bijection φ of the space $\mathrm{Vect}_\mathcal{M}$ onto some $C^\infty(M)$-module $\Lambda^1(\mathcal{M}) \subset \Lambda^1(M)$ of differential forms on the blow-up M. (To see this, it suffices to note that the embedding $\mathrm{Vect}_M \subset \mathrm{Vect}_\mathcal{M}$ is dense on the main stratum.)

Definition 2.4. The *cotangent bundle* $T^*\mathcal{M}$ is the vector bundle (existing by Swan's theorem) over M whose space of sections is $\Lambda^1(\mathcal{M})$.

Remark 2.2. Note that the elements of $\Lambda^1(\mathcal{M})$ are precisely the forms vanishing on the fibers of the projection $\pi : M \longrightarrow \mathcal{M}$.

Remark 2.3. Since differential operators on M are just polynomials in vector fields $X \in \mathrm{Vect}_\mathcal{M}$ with smooth ($C^\infty(M)$) coefficients, one can readily show that their interior symbols are smooth functions on $T^*\mathcal{M}$.

The cotangent bundles of the strata $\mathcal{M}_j$ are defined similarly.

The space $C^\infty(\mathcal{M})$. By definition, the elements of this space are smooth functions on the blow-up M of $\mathcal{M}$ that depend only on x for small r in the coordinates (x, r, ω) in a neighborhood of each stratum of nonmaximal dimension.

2.2 Pseudodifferential operators and symbols

Here we describe an algebra of zero-order ψDO on a stratified manifold. More precisely, we define families of ψDO with parameter v ranging in a finite-dimensional vector space V; by successive trivial generalizations at each inductive step, we also obtain ψDO smoothly depending on some additional parameters x and then ψDO parametrized by points of a finite-dimensional vector bundle over a smooth manifold. Details of these generalizations are left to the reader.

Negligible families. We introduce the space of operator families modulo which ψDO will be defined below. Let $\mathcal{M}$ be a stratified manifold (possibly, noncompact). Let $J_\infty(V,\mathcal{M}) \equiv J_\infty(\mathcal{M}) \equiv J_\infty$ be the space of smooth operator families

$$D(v) : L^2(\mathcal{M}) \longrightarrow L^2(\mathcal{M}), \quad v \in V, \tag{4}$$

such that all operators $D(v)$ are compact in $L^2(\mathcal{M})$ and satisfy the estimates

$$\left\| \frac{\partial^\beta D(v)}{\partial v^\beta} \right\| \le C_{\beta N}(1+|v|)^{-N}, \qquad |\beta|, N = 0,1,2,\dots, \tag{5}$$

and these conditions hold if we replace $D(v)$ by the product

$$V_1 \cdots V_p D(v) V_{p+1} \cdots V_{p+q}$$

of arbitrary length $p+q \ge 0$. Here $V_1,\dots,V_{p+q}$ are smooth vector fields on M that have the form $V = (V(x), 0, \widetilde{V}(x,\omega))$ in local coordinates $(x,r,\omega) \in \mathbb{R}^m \times \overline{\mathbb{R}}_+ \times \Omega$ near each stratum, where $V(x)$ is a smooth vector field on the stratum and $\widetilde{V}(x,\omega)$ is a vector field, smoothly depending on $x \in X$, with the same properties on the manifold Ω of lower dimension.

ψDO on smooth manifolds. We are now in a position to define ψDO. Our definition is by induction. We start from the class of families of ψDO on smooth manifolds.

Definition 2.5. An operator family

$$D(v) : L^2(\mathcal{M}) \longrightarrow L^2(\mathcal{M})$$

on a smooth manifold $\mathcal{M}$ is called a *pseudodifferential operator with parameter* $v \in V$ if it is a zero-order ψDO on $\mathcal{M}$ with parameter $v \in V$ in the sense of Agranovich–Vishik. (These are often called parameter-dependent ψDO.)

The *symbol* (corresponding to the unique stratum of $\mathcal{M}$) is by definition the symbol $\sigma(D)(x,\xi,v)$ in the sense of Agranovich–Vishik. It is defined on the total space of the vector bundle $T^*\mathcal{M} \times V$ over $\mathcal{M}$ minus the zero section.

The space of pseudodifferential operators with parameter $v \in V$ on $\mathcal{M}$ is denoted by $\Psi(V,\mathcal{M})$. (In what follows, V will be suppressed if it is clear from the context or trivial.)

ψDO on stratified manifolds. Here we define ψDO by induction over the length k of the stratified manifold. Note that we simultaneously define ψDO and their symbols. Definition 2.5 will serve as the inductive statement for $k=0$.

Definition 2.6. Let $\mathcal{M}$ be a stratified manifold of length $k>0$. A smooth family of linear operators

$$D(v): L^2(\mathcal{M}) \longrightarrow L^2(\mathcal{M})$$

is called a *pseudodifferential operator on* $\mathcal{M}$ (*with parameter* $v \in V$ *in the sense of Agranovich–Vishik*) if the following conditions hold.

1. Given $\varphi, \psi \in C^\infty(\mathcal{M})$ such that $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \varnothing$, we have $\psi A \varphi \in J_\infty$.
2. D is a ψDO with parameter on[3] $\mathcal{M} \setminus \mathcal{M}_0$; in a neighborhood U of $\mathcal{M}_0$, the operator D is representable in the form

$$D = P\biggl(x, r, rv, -ir\frac{\partial}{\partial x}, ir\frac{\partial}{\partial r} + i\frac{n+1}{2}\biggr), \qquad n = \dim\Omega \tag{6}$$

modulo the ideal $J_\infty(V, \mathcal{M})$, where $P(x, r, v, \eta, p) \in \Psi(V \times T^*_x\mathcal{M}_0 \times \mathbb{R}, \Omega)$ is a ψDO with parameters on Ω, depending smoothly on additional parameters $x \in \mathcal{M}_0$ and $r \in \overline{\mathbb{R}}_+$; moreover, $P = 0$ for $r > r_0$, where r_0 is small.

By definition, the symbols $\sigma_j(D)$ of D corresponding to the strata $\mathcal{M}_j \setminus \mathcal{M}_{j-1}$, $j>0$, of $\mathcal{M}$, are its symbols as an element of $\Psi(V, \mathcal{M} \setminus \mathcal{M}_0)$. The *symbol of D corresponding to the stratum* $\mathcal{M}_0$ is the operator family

$$\sigma_0(D) = P\biggl(x, 0, rv, r\xi, ir\frac{\partial}{\partial r} + i\frac{n+1}{2}\biggr) : L^2(K_\Omega) \longrightarrow L^2(K_\Omega) \tag{7}$$

parametrized by the points of the bundle $V \times T^*\mathcal{M}_0$ over $\mathcal{M}_0$ minus the zero section.

The symbols $\sigma_j(\sigma_0(D))$ of the symbol $\sigma_0(D)$ are just the symbols $\sigma_j(P(x,0,v,\eta,p))$ of the ψDO $P(x,0,v,\eta,p)$ with parameters on Ω, $j = 1, \dots, k$.

Let us give some explanations concerning formulas (6) and (7). One can prove by induction over the length of $\mathcal{M}$ that a pseudodifferential operator $A \in \Psi(V, \mathcal{M})$ satisfies the estimates

$$\left\| \frac{\partial^\alpha A(v)}{\partial v^\alpha} \right\| \le C_\alpha (1+|v|)^{-|\alpha|}, \qquad |\alpha| = 0, 1, 2, \dots, \tag{8}$$

where all derivatives starting from the first are compact-valued. Indeed, using this fact for $\mathcal{M} = \Omega$, we see that the operator family

$$F(x, t, v, \xi, p) = P(x, e^{-t}, ve^{-t}, \xi e^{-t}, p)$$

satisfies the estimates

$$\left\| \frac{\partial F(x,t,v,\xi,p)}{\partial x^\alpha \partial t^l \partial v^\beta \partial \xi^\gamma \partial p^k} \right\| \le C_{\alpha l \beta \gamma k} (1 + |v| + |\xi|)^{-|\beta| - |\gamma|} (1+|p|)^{-k},$$
$$|\alpha| + l + |\beta| + |\gamma| + k = 0, 1, 2, \dots, \tag{9}$$

[3] The stratum $\mathcal{M}_0$ has measure zero in $\mathcal{M}$. Hence D is automatically interpreted as an operator on $L^2(\mathcal{M} \setminus \mathcal{M}_0)$.

and the operator family

$$\widetilde{F}(x,t,v,\xi,p) = P(x,0,ve^{-t},\xi e^{-t},p)$$

satisfies the estimates

$$\begin{aligned}\left\|\frac{\partial \widetilde{F}(x,t,v,\xi,p)}{\partial x^\alpha \partial t^l \partial v^\beta \partial \xi^\gamma \partial p^k}\right\| &\le C_{\alpha l\beta\gamma k}(e^t+|v|+|\xi|)^{-|\beta|-|\gamma|}(1+|p|)^{-k} \\ &\le C_{\alpha l\beta\gamma k}(|v|+|\xi|)^{-|\beta|-|\gamma|}(1+|p|)^{-k}, \quad |\alpha|+l+|\beta|+|\gamma|+k=0,1,2,\dots. \end{aligned} \tag{10}$$

Moreover, both families have compact variation in the parameters (v,ξ,p). It is not difficult to show that the operators on the right-hand sides in (6) and (7) are well defined as ψDO in the sense of Luke [18]. Indeed, the change of variable $r=e^{-t}$ transforms the cone K_Ω into the cylinder $\Omega\times\mathbb{R}$ and the operator $ir\partial/\partial r$ into $-i\partial/\partial t$. It remains to note that the operator $-i\partial/\partial t + i(n+1)/2$ is self-adjoint in the L^2 space with the weight $e^{-(n+1)t}$ on the cylinder. This space is just the image of $L^2(K_\Omega)$ under our change of variables. Therefore, the substitution of this operator as an operator argument into (7) is well defined. In addition, the constructed ψDO satisfies the estimates (8). This ends the inductive step.

Remark 2.4. Since the cone is noncompact (in r), the operator-valued symbol (7) only has *almost compact* variation in (ξ,v) in the general case. (That is, the variation becomes compact if we multiply it by a cut-off function finite in r). However, as we shall see shortly, the fiber variation of the symbol in (7) is compact provided that all symbols $\sigma_j(\sigma_0(D))$ are zero, $j=1,\dots,k$.

Definition 2.7. The *conormal symbol* $\sigma_c(\sigma_0(D))\in\Psi(\mathcal{M}_0^\circ\times V\times\mathbb{R},\Omega)$ of the family (7) is the family $\sigma_c(\sigma_0(D))=P(x,0,v,0,p)$.

Compatibility conditions. We have defined the notion of ψDO with parameters on a stratified manifold $\mathcal{M}$. Such ψDO have symbols defined a priori on the cotangent bundles of the *open strata* times the parameter space minus the zero section. However, if we compare the representation (6) with the representation valid in $U\setminus\mathcal{M}_0$ by the inductive statement, then we can see that actually the symbols extend continuously (and smoothly) up to the boundary of the cotangent bundle and hence are defined on the cotangent bundles of the corresponding *closed strata*. Moreover, at the points where a stratum $\mathcal{M}_j$ meets a stratum $\mathcal{M}_i$, $j>i$, the following compatibility conditions hold:

$$\sigma_l(\sigma_j(D))|_{\mathcal{M}_i}=\sigma_l(\sigma_i(D)), \qquad l=j,\dots,k. \tag{11}$$

Here we write $\sigma_j(\sigma_j(D))=\sigma_j(D)$ for brevity.

Main properties of the calculus of ψDO. Let us state the main properties of the calculus of ψDO on a compact stratified manifold. The proofs (except for the proof of Proposition 2.2) are omitted; they can be found in [19].

The set of all symbols (7) on $X=\mathcal{M}_0$ is denoted by $\Psi(T^*X\times V,K_\Omega)$.

Proposition 2.1. *$\Psi(T^*X \times V, K_\Omega)$ is a local C^*-algebra.*

The norm is given by the supremum of the operator norm over all parameter values.

Theorem 2.1 (Main properties of ψDO). *Pseudodifferential operators enjoy the following properties.*

(1) *$\Psi(V, \mathcal{M})$ is an algebra with the usual composition of operators and is a local C^*-algebra with respect to the supremum of the operator norm over the parameter space. Pseudodifferential operators compactly commute with the operators of multiplication by continuous functions on $\mathcal{M}$.*

(2) *The symbol map*

$$\begin{aligned} \sigma : \Psi(V, \mathcal{M}) &\longrightarrow \bigoplus_{j=0}^{k} \Psi(T^*\mathcal{M}_j \times V, K_{\Omega_j}), \\ D &\longmapsto (\sigma_0(D), \dots, \sigma_k(D)) \end{aligned} \tag{12}$$

is a local C^*-algebra homomorphism and induces an isomorphism

$$\sigma : \Psi(V, \mathcal{M}) \,/\, J(V, \mathcal{M}) \longrightarrow \Sigma(V, \mathcal{M}) \subset \bigoplus_{j=0}^{k} \Psi(T^*\mathcal{M}_j \times V, K_{\Omega_j})$$

onto the local C^*-algebra of symbols subject to the compatibility conditions (11). *Here $J(V, \mathcal{M}) \subset \Psi(V, \mathcal{M})$ denotes the ideal of compact-valued operator families vanishing at infinity.*

(3) *$\Psi(V, \mathcal{M})$ is invariant under diffeomorphisms of $\mathcal{M}$.*

Definition 2.8. An operator $D \in \Psi(V, \mathcal{M})$ is said to be *elliptic* if all its symbols $\sigma_j(D)$, $j = 0, \dots, k$, are invertible in the complements of the zero sections of the corresponding bundles.

As a corollary of Theorem 2.1, we obtain the following assertion

Theorem 2.2. (1) *On a compact manifold $\mathcal{M}$, elliptic operators are Fredholm* (*for all parameter values*)*.*

(2) *If $V \neq \{0\}$, then an elliptic operator with parameter is invertible for large $|v|$.*

The following proposition will be useful.

Proposition 2.2. *Let $\Sigma_0 \subset \Psi(V, K_\Omega)$ be the set of symbols whose symbols corresponding to the strata of $\Omega \times \mathbb{R}_+$ are zero. Then each symbol $\sigma \in \Sigma_0$ has compact variation in v.*

Proof. By the definition (see Eq. (7)),

$$\sigma(v) = P\left(rv, ir\frac{\partial}{\partial r} + i\frac{n+1}{2}\right) : L^2(K_\Omega) \longrightarrow L^2(K_\Omega)$$

for an operator function $P(w,p) \in \Psi(V \times \mathbb{R}, \Omega)$. Theorem 2.1 implies that $P(w,p) \in J(V \times \mathbb{R}, \Omega)$ and $P(w,p)$ satisfies the estimates (8) with respect to (w,p).

We have to show that, given $v \neq 0$, $\partial\sigma/\partial v$ is compact. We have

$$\frac{\partial \sigma}{\partial v} = r\frac{\partial P}{\partial w}\left(rv, ir\frac{\partial}{\partial r} + i\frac{n+1}{2}\right).$$

The symbol $r\partial P/\partial w(rv,p)$ possesses all estimates needed for it to define a bounded operator on $L^2(K_\Omega)$ (cf. (10)). Furthermore, it is compact-valued. It remains to show that it tends to zero as $r \to 0$ or $r \to \infty$. For $r \to 0$, this is obvious (by virtue of the factor r), and for $r \to \infty$ we use the representation

$$r\frac{\partial P}{\partial w}(rv,p) = \frac{1}{|v|}|rv|\frac{\partial P}{\partial w}(rv,p) = \frac{1}{|v|}\left[|w|\frac{\partial P}{\partial w}\right]_{w=rv}$$

and apply the following lemma to $P(w,p)$.

Lemma 2.1. *Let $f \in C^2(\mathbb{R}^n)$ be an operator function such that $f(\xi) \to 0$ as $\xi \to \infty$ and $|f''(\xi)| \leq C|\xi|^{-2}$ for large ξ. Then $|\xi||f'(\xi)| \to 0$ as $\xi \to \infty$.*

□

Semiclassical quantization. Consider the quantization (7) with a “semiclassical” parameter h:

$$T_h : \Psi(T^*X \times \mathbb{R}, \Omega) \longrightarrow \Psi(T^*X, K_\Omega) \subset \mathcal{B}(L^2(K_\Omega)).$$

This map takes a family with parameter on the base of the cone to a family of operators on the infinite cone and is defined as

$$(T_h D)(\xi) := D\left(\overset{2}{r}\xi, ihr\overset{1}{\frac{\partial}{\partial r}} + ih(n+1)/2\right). \tag{13}$$

By the same argument as in [7, Appendix], it can be proved that this quantization is asymptotic in L^2, i.e., satisfies the estimates

$$T_h(a)T_h(b) = T_h(ab) + o(1), \quad h \to 0, \tag{14}$$

in the sense of operator norm.

We shall use this quantization below in the computation of the boundary map in K-theory of ψDO algebras.

3 Ell-groups

From now on, we fix a compact stratified manifold $\mathcal{M}$.

Definition 3.1. Two elliptic operators

$$D : L^2(\mathcal{M}, E) \to L^2(\mathcal{M}, F) \text{ and } D' : L^2(\mathcal{M}, E') \to L^2(\mathcal{M}, F')$$

acting between sections of vector bundles over the blow-up M are said to be *stably homotopic* if there exists a continuous homotopy [4] of elliptic operators

$$D \oplus 1_{E_0} \sim f^*\big(D' \oplus 1_{F_0}\big)e^*,$$

where $E_0, F_0 \in \mathrm{Vect}(M)$ are vector bundles and

$$e : E \oplus E_0 \longrightarrow E' \oplus F_0, \qquad f : F' \oplus F_0 \longrightarrow F \oplus E_0$$

are vector bundle isomorphisms.

Here ellipticity means the invertibility of symbol components on all the strata (see Def. 2.8), and we consider only homotopies of ψDO preserving ellipticity.

Even groups $\mathrm{Ell}_0(\mathcal{M})$. Stable homotopy is an equivalence relation on the set of all elliptic pseudodifferential operators acting between vector bundle sections. Let $\mathrm{Ell}_0(\mathcal{M})$ be the quotient by this equivalence relation. This set is a group with respect to the direct sum of elliptic operators. The inverse is defined by the almost inverse operator (that is, inverse modulo compact operators).

Odd groups $\mathrm{Ell}_1(\mathcal{M})$. We can similarly define the odd elliptic theory $\mathrm{Ell}_1(\mathcal{M})$ as the group of stable homotopy classes of elliptic self-adjoint operators. In this case, the stabilization is done in terms of the operators $\pm Id$.

Remark 3.1. An equivalent definition of the odd Ell-group can be given in terms of elliptic families on $\mathcal{M}$ parametrized by the circle $\mathbb{S}^1$ modulo constant families.

We consider the following *homotopy classification problem for elliptic operators:* compute the group $\mathrm{Ell}_*(\mathcal{M})$.

4 Main theorem

The map into K-homology. Let

$$D : L^2(\mathcal{M}, E) \longrightarrow L^2(\mathcal{M}, F)$$

be an elliptic operator. By Theorems 2.2 and 2.1, (1), this operator can be treated as an abstract elliptic operator in the sense of Atiyah [1] on $\mathcal{M}$. Thus it defines an element in the K-homology of $\mathcal{M}$. The corresponding Fredholm module is defined in the standard way. Namely (cf. [20]), if D is self-adjoint (and $E = F$), then consider the normalization

$$\mathcal{D} = (P_{\ker D} + D^2)^{-1/2} D : L^2(\mathcal{M}, E) \longrightarrow L^2(\mathcal{M}, E), \tag{15}$$

where $P_{\ker D}$ is the projection onto the null space of D.

In the general case, we consider the self-adjoint operator

$$\mathcal{D} = \begin{pmatrix} 0 & D(P_{\ker D} + D^*D)^{-1/2} \\ (P_{\ker D} + D^*D)^{-1/2} D^* & 0 \end{pmatrix} : L^2(\mathcal{M}, E \oplus F) \to L^2(\mathcal{M}, E \oplus F), \tag{16}$$

which is odd with respect to the $\mathbb{Z}_2$-grading of $L^2(\mathcal{M}, E) \oplus L^2(\mathcal{M}, F)$.

[4] We can and will always assume that the homotopies are such that all symbols and their derivatives are continuously homotopic in the corresponding norms.

Proposition 4.1. *1. The operators* (15) *and* (16) *define K-homology elements denoted by $[D] \in K_*(\mathcal{M})$, where $*=1$ in the self-adjoint case and $*=0$ otherwise.*

2. There is a well-defined group homomorphism

$$\begin{array}{ccc} \mathrm{Ell}_*(\mathcal{M}) & \xrightarrow{\varphi} & K_*(\mathcal{M}), \\ D & \mapsto & [D]. \end{array}$$

Proof. The operators $\mathcal{D}$ in (15) and (16) are self-adjoint and act in $*$-modules over the C^*-algebra $C(\mathcal{M})$. To end the proof of the first part of the proposition, it suffices to check that, given $f \in C(\mathcal{M})$, one has

$$[\mathcal{D}, f] \in \mathcal{K}, \quad (\mathcal{D}^2 - 1)f \in \mathcal{K}, \tag{17}$$

where $\mathcal{K}$ is the ideal of compact operators. The compactness follows easily from the composition formula for pseudodifferential operators (since $\mathcal{D}$ is a pseudodifferential operator). The map is well defined, since homotopies of elliptic operators give continuous homotopies of the corresponding Fredholm modules, i.e., give the same K-homology element. Bundle isomorphisms give degenerate Fredholm modules. (Recall [20] that a module is degenerate if all expressions in (17) are zero.) $\square$

The classification theorem. The following theorem solves the classification problem on stratified manifolds.

Theorem 4.1. *The map*

$$\mathrm{Ell}_*(\mathcal{M}) \overset{\varphi}{\simeq} K_*(\mathcal{M})$$

that takes each elliptic operator D to the element defined in Proposition 4.1 *is an isomorphism.*

The nondegeneracy of the index pairing $K_0(\mathcal{M}) \times K^0(\mathcal{M}) \longrightarrow \mathbb{Z}$ (on the torsion-free parts of the groups) gives the following assertion.

Corollary 4.1. *Two elliptic operators D_1 and D_2 are stably rationally homotopic if and only if their indices with coefficients in any vector bundle over $\mathcal{M}$ are equal.*

We shall obtain Theorem 4.1 as a special case of a more general theorem, which we now state.

The classification of partially elliptic operators. An operator D on $\mathcal{M}$ is said to be *elliptic on* $\mathcal{M} \backslash \mathcal{M}_j$ if the symbol components $\sigma_k(D), \ldots, \sigma_{j+1}(D)$ are invertible on their domains, i.e., everywhere on the complement of the zero section in $T^*\mathcal{M}_k, \ldots, T^*\mathcal{M}_{j+1}$.

Let $\mathrm{Ell}_*(\mathcal{M}, \mathcal{M}_j)$ be the group of stable homotopy classes of pseudodifferential operators elliptic on $\mathcal{M} \setminus \mathcal{M}_j$. We assume that the homotopies are also taken in this class.

By analogy with the map φ in Proposition 4.1, we define

$$\mathrm{Ell}_*(\mathcal{M}, \mathcal{M}_j) \xrightarrow{\varphi} K_*(\mathcal{M} \setminus \mathcal{M}_j).$$

There is one important difference: to define φ in this case, we replace $(P_{\ker D}+D^*D)^{-1/2}$ in (15) and (16) by self-adjoint pseudodifferential operators, with leading $k-j$ symbol components equal to

$$(\sigma_k(D)^*\sigma_k(D))^{-1/2},\dots,(\sigma_{j+1}(D)^*\sigma_{j+1}(D))^{-1/2}.$$

Note that the two constructions give the same K-homology element for an operator elliptic on the entire $\mathcal{M}$.

Theorem 4.2. *Given $-1\le j\le k-1$, one has the isomorphism*

$$\mathrm{Ell}_*(\mathcal{M},\mathcal{M}_j)\overset{\varphi}{\simeq}K_*(\mathcal{M}\setminus\mathcal{M}_j).$$

The proof of Theorem 4.2 occupies Sections 5–7. First, in Section 5, Ell-groups are represented as K-groups of certain algebras (noncommutative analog of the Atiyah–Singer difference construction). This enables us to define exact sequences for Ell-groups. Then the proof is done inductively over the strata in Sections 6 and 7.

5 Relationship between Ell-groups and K-theory

Ell**-groups as K-groups of C^*-algebras**. The embedding

$$C^\infty(M)\subset\Psi(\mathcal{M})$$

(the embedding corresponds to the usual action of functions as multiplication operators) of algebras of scalar operators enables one to describe pseudodifferential operators in vector bundle sections. Namely, an arbitrary zero-order ψDO acting between spaces of vector bundle sections can be represented as

$$D':\operatorname{Im}P\longrightarrow\operatorname{Im}Q,$$

where P and Q are matrix projections ($P^2=P,Q^2=Q$) with entries in $C^\infty(M)$ and D' is a matrix operator with entries in $\Psi(\mathcal{M})$.

Denote by

$$\Sigma(\mathcal{M}\setminus\mathcal{M}_j)\overset{\mathrm{def}}{=}\operatorname{Im}(\sigma_k,\dots,\sigma_{j+1})\subset\bigoplus_{l\ge j+1}C^\infty(S^*M_l,\mathcal{B}(L^2(K_{\Omega_l})))$$

the algebra of leading $k-j$ components of the principal symbol.

Theorem 4 in [6] gives isomorphisms

$$\mathrm{Ell}_*(\mathcal{M},\mathcal{M}_j)\overset{\chi}{\simeq}K_*(\mathrm{Con}(C^\infty(M)\xrightarrow{f}\Sigma(\mathcal{M}\setminus\mathcal{M}_j)))\qquad(18)$$

of the Ell-groups and the K-groups of special local C^*-algebras. Here

$$f:C^\infty(M)\longrightarrow\Sigma(\mathcal{M}\setminus\mathcal{M}_j)$$

is the embedding taking a smooth function on the blow-up M to the symbol of multiplication operator by this function, and

$$\mathrm{Con}(A \xrightarrow{f} B) = \Big\{(a, b(t)) \in A \oplus C_0([0,1), B) \mid f(a) = b(0)\Big\}$$

is the mapping cone of the algebra homomorphism $f : A \to B$.

In most cases, one can eliminate the mapping cone from Eq. (18) by using the following lemma.

Lemma 5.1. *The K-group of the mapping cone can be represented as*

$$K_{*+1}(\mathrm{Con}(C^\infty(M) \to \Sigma(\mathcal{M} \setminus \mathcal{M}_j))) \simeq K_*(\Sigma(\mathcal{M} \setminus \mathcal{M}_j))/K_*(C^\infty(M)),$$

provided that M has a nonsingular vector field. (*This condition is satisfied if, say, M has no components with empty boundary.*)

Proof. A vector field $M \to S^*M$ defines a section $\Sigma(\mathcal{M} \setminus \mathcal{M}_j) \to C^\infty(M)$. Thus the mapping cone exact sequence (induced by the embedding of algebras) splits. The splitting gives the desired isomorphism. $\square$

Remark 5.1. In the odd case, the composition of the latter isomorphism with χ shows that, modulo stable homotopy, elliptic self-adjoint operators are isomorphic to symbols-projections modulo projections onto bundle sections (cf. [21]).

Exact sequence of the pair in Ell**-theory**. Let us construct the exact sequence corresponding in elliptic theory to the pair

$$\mathcal{M}_j \setminus \mathcal{M}_{j-1} \subset \mathcal{M} \setminus \mathcal{M}_{j-1}.$$

We define the maps in the desired sequence in K-theoretic terms. To this end, consider the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \ker(\sigma_k, \dots, \sigma_{j+1}) & \longrightarrow & \Sigma(\mathcal{M} \setminus \mathcal{M}_{j-1}) & \overset{(\sigma_k,\dots,\sigma_{j+1})}{\longrightarrow} & \Sigma(\mathcal{M} \setminus \mathcal{M}_j) & \to 0 \\ & \uparrow & & \uparrow & & \uparrow & \\ 0 \to & 0 & \longrightarrow & C^\infty(M) & = & C^\infty(M) & \to 0 \end{array} \qquad (19)$$

with exact rows. The ideal $\ker(\sigma_k, \dots, \sigma_{j+1})$ of symbols with leading $k - j$ components equal to zero is denoted for brevity by Σ_0. The diagram induces the exact sequence

$$0 \to S\Sigma_0 \to \mathrm{Con}(C^\infty(M) \to \Sigma(\mathcal{M} \setminus \mathcal{M}_{j-1})) \to \mathrm{Con}(C^\infty(M) \to \Sigma(\mathcal{M} \setminus \mathcal{M}_j)) \to 0 \quad (20)$$

of the mapping cones of the vertical embeddings. Here $S\Sigma_0 = C_0((0,1), \Sigma_0)$ stands for suspension.

The K-groups of the mapping cones in (20) classify elliptic operators on $\mathcal{M} \setminus \mathcal{M}_{j-1}$ and $\mathcal{M} \setminus \mathcal{M}_j$, respectively (see Eq. (18)). It turns out that the K-groups of the ideal also classify elliptic operators.

Lemma 5.2. *The K-groups of $S\Sigma_0$ classify operators that are elliptic on $\mathcal{M} \setminus \mathcal{M}_{j-1}$ and have symbols $(\sigma_k, \dots, \sigma_{j+1})$ induced by constant functions in $C^\infty(M)$.*

Proof. To be definite, we consider $K_0(S\Sigma_0)$. Here $K_0(S\Sigma_0) = K_1(\Sigma_0) = K_1(\Sigma_0^+)$, where Σ_0^+ denotes the algebra with identity adjoined. Now it follows from the definition of K_1 and Σ_0 that $K_1(\Sigma_0^+)$ classifies elliptic operators described in the lemma. $\square$

Using this lemma, we can define a map (cf. Proposition 4.1)

$$\varphi : K_*(S\Sigma_0) \to K_*(\mathcal{M}_j \setminus \mathcal{M}_{j-1}) \tag{21}$$

that restricts elliptic operators to a neighborhood U of the subset $\mathcal{M}_j \setminus \mathcal{M}_{j-1}$. We assume that U has the structure of a bundle $\pi : U \to \mathcal{M}_j \setminus \mathcal{M}_{j-1}$ with conical fiber. Note that the structure of a $C_0(\mathcal{M}_j \setminus \mathcal{M}_{j-1})$-module

$$C_0(\mathcal{M}_j \setminus \mathcal{M}_{j-1}) \xrightarrow{\pi^*} C(U) \to \mathcal{B}(L^2(U))$$

on the corresponding L^2-spaces where the operator acts is obtained by the pull-back from the base of the bundle. The restriction of the operator is well defined, since in the complement of an arbitrarily small neighborhood of $\mathcal{M}_j \setminus \mathcal{M}_{j-1}$ the operator acts as the multiplication by some function.

We shall show in Lemma 7.1 that

$$K_*(S\Sigma_0) \simeq K_c^*(T^*M_j) = \mathrm{Ell}_*(\mathcal{M}_j, \mathcal{M}_{j-1}). \tag{22}$$

Therefore, we can replace all K-groups by Ell-groups in the exact sequence induced in K-theory by (20). As a result, we obtain the periodic six-term exact sequence relating the Ell-groups:

$$\begin{tikzcd}[column sep=small]
 & \mathrm{Ell}_0(\mathcal{M}_j, \mathcal{M}_{j-1}) \arrow[r] & \mathrm{Ell}_0(\mathcal{M}, \mathcal{M}_{j-1}) \arrow[dr] & \\
\mathrm{Ell}_1(\mathcal{M}, \mathcal{M}_j) \arrow[ur, "\partial"] & & & \mathrm{Ell}_0(\mathcal{M}, \mathcal{M}_j) \arrow[dl, "\partial"] \\
 & \mathrm{Ell}_1(\mathcal{M}, \mathcal{M}_{j-1}) \arrow[ul] & \mathrm{Ell}_1(\mathcal{M}_j, \mathcal{M}_{j-1}) . \arrow[l] &
\end{tikzcd} \tag{23}$$

We do not dwell upon the question of defining all maps in this sequence directly in terms of elliptic operators.

6 Induction

For j ranging from k down to -1, we shall prove by induction that the map φ in Theorem 4.2 is an isomorphism on the set $\mathcal{M} \setminus \mathcal{M}_j$.

For $j = k$, this is obvious. Let us prove the inductive step: the isomorphism for $j+1$ implies an isomorphism for j.

The pair $\mathcal{M}_j \setminus \mathcal{M}_{j-1} \subset \mathcal{M} \setminus \mathcal{M}_{j-1}$ gives the diagram

$$\begin{array}{ccccccccc}
\ldots \to \mathrm{Ell}_{*+1}(\mathcal{M}, \mathcal{M}_j) & \xrightarrow{\partial} & \mathrm{Ell}_*(\mathcal{M}_j, \mathcal{M}_{j-1}) & \to & \mathrm{Ell}_*(\mathcal{M}, \mathcal{M}_{j-1}) & \to & \mathrm{Ell}_*(\mathcal{M}, \mathcal{M}_j) & \xrightarrow{\partial} & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots \to K_{*+1}(\mathcal{M} \setminus \mathcal{M}_j) & \xrightarrow{\partial} & K_*(\mathcal{M}_j \setminus \mathcal{M}_{j-1}) & \to & K_*(\mathcal{M} \setminus \mathcal{M}_{j-1}) & \to & K_*(\mathcal{M} \setminus \mathcal{M}_j) & \xrightarrow{\partial} & \ldots
\end{array} \tag{24}$$

Note that the vertical maps are defined in Proposition 4.1 and Eqs. (21) and (22). Once we prove that the diagram commutes, the inductive assumption, in conjuntion with the 5-lemma, will imply that the vertical map ranging in $K_*(\mathcal{M}\setminus\mathcal{M}_{j-1})$ is an isomorphism, as desired. Let us establish the commutativity.

The commutativity of the square

$$\begin{array}{ccc} \mathrm{Ell}_*(\mathcal{M},\mathcal{M}_{j-1}) & \longrightarrow & \mathrm{Ell}_*(\mathcal{M},\mathcal{M}_j) \\ \downarrow & & \downarrow \\ K_*(\mathcal{M}\setminus\mathcal{M}_{j-1}) & \longrightarrow & K_*(\mathcal{M}\setminus\mathcal{M}_j) \end{array}$$

is obvious, since the horizontal maps are induced by forgetting some of the structures.

The commutativity of the square

$$\begin{array}{ccc} \mathrm{Ell}_*(\mathcal{M}_j,\mathcal{M}_{j-1}) & \longrightarrow & \mathrm{Ell}_*(\mathcal{M},\mathcal{M}_{j-1}) \\ \downarrow & & \downarrow \\ K_*(\mathcal{M}_j\setminus\mathcal{M}_{j-1}) & \longrightarrow & K_*(\mathcal{M}\setminus\mathcal{M}_{j-1}), \end{array}$$

corresponding to the embedding $i:\mathcal{M}_j\setminus\mathcal{M}_{j-1}\longrightarrow\mathcal{M}\setminus\mathcal{M}_{j-1}$, is also easy to prove. Indeed, consider the composition of the maps through the top right corner of the square: it takes an operator elliptic on $\mathcal{M}\setminus\mathcal{M}_{j-1}$ and equal to a multiplication operator in the complement of a neighborhood U of $\mathcal{M}_j\setminus\mathcal{M}_{j-1}$ to the same operator with the natural $C_0(\mathcal{M}\setminus\mathcal{M}_{j-1})$-module structure on the spaces between which the operator acts. We restrict the operator to a neighborhood of $\mathcal{M}_j\setminus\mathcal{M}_{j-1}$ (this does not change the K-homology element, since the corresponding Fredholm module changes by a degenerate module) and make a homotopy of the module structure to the composition

$$C_0(\mathcal{M}\setminus\mathcal{M}_{j-1})\xrightarrow{i^*}C_0(\mathcal{M}_j\setminus\mathcal{M}_{j-1})\xrightarrow{\pi^*}C(U)\to\mathcal{B}(L^2(U)),$$

where $\pi:U\to\mathcal{M}_j\setminus\mathcal{M}_{j-1}$ is the projection. This gives the very same element as the composition of arrows through the bottom left corner of the square. This proves the commutativity of the square.

To justify the inductive step, it remains to show the commutativity of the squares containing the boundary maps in (24). This is the most technically involved part of the proof. We do this in the next section.

7 Comparison of boundary maps

In this section, we prove the commutativity of the squares containing the boundary maps in (24). The proof is carried out in terms of K-groups of symbol algebras. The proof is rather long; therefore, here we list the main steps:

- We show that the K-homology boundary map is the composition of the restriction to the boundary of the stratum and the direct image (see Subsec. 1 below).

- The K-theory boundary map is also the composition of the restriction map and a certain boundary map ∂' related to the algebra $\Psi(T^*M_j,K_\Omega)$ of symbols on M_j (Subsec. 2). Hence the comparison of the boundary maps in K-theory and K-homology is reduced to M_j.

- Unfortunately, the boundary map ∂' is not so easy to work with. Therefore, we replace $\Psi(T^*M_j, K_\Omega)$ by the simpler algebra $\Psi(T^*M_j \times \mathbb{R}, \Omega)$ of families with parameters. It turns out that the boundary map ∂'' for the latter algebra is easy do compute. More precisely, in Subsec. 3 we define an asymptotic homomorphism of one algebra into the other, and in Subsec. 4 we show that the asymptotic homomorphism induces an isomorphism on K-groups.

- The boundary map ∂'' is computed in Subsec. 5 in terms of the families index of an elliptic family with parameters. Then the compatibility of the boundary map ∂'' and the direct image in K-homology is verified in Theorem 7.1 in Subsec. 5. This finishes the proof of the commutativity in (24).

Let us now give the details of the proof.

1. The boundary map in K-homology in (24). We start with some notation. Let U be a neighborhood of $\mathcal{M}_j \setminus \mathcal{M}_{j-1} = \mathcal{M}_j^\circ$ fibered with conical fiber over the stratum, and let $\pi : U \to \mathcal{M}_j \setminus \mathcal{M}_{j-1}$ be the corresponding projection. The bundle of cone bases is denoted by

$$\pi' : \Xi \longrightarrow \mathcal{M}_j \setminus \mathcal{M}_{j-1}.$$

The typical fiber is Ω. One obviously has $U \setminus \mathcal{M}_j^\circ \simeq \mathbb{R}_+ \times \Xi$.

We claim that the boundary map $\partial : K_*(\mathcal{M} \setminus \mathcal{M}_j) \to K_{*+1}(\mathcal{M}_j \setminus \mathcal{M}_{j-1})$ is equal to the composition

$$K_*(\mathcal{M} \setminus \mathcal{M}_j) \longrightarrow K_*(U \setminus \mathcal{M}_j^\circ) = K_{*+1}(\Xi) \xrightarrow{\pi'_*} K_{*+1}(\mathcal{M}_j \setminus \mathcal{M}_{j-1}) \qquad (25)$$

of the restriction to $U \setminus \mathcal{M}_j^\circ$, the periodicity isomorphism, and the direct image map. This decomposition is easy to obtain by using the fact that the boundary map is natural.

2. Reduction to the boundary. Let us start the computation of the boundary map in K-theory of algebras. First, being natural, the boundary map

$$\partial : K_*(\mathrm{Con}(C^\infty(M) \to \Sigma(\mathcal{M} \setminus \mathcal{M}_j))) \to K_{*+1}(S\Sigma_0)$$

is equal to the composition

$$K_*(\mathrm{Con}(C^\infty(M) \to \Sigma(\mathcal{M} \setminus \mathcal{M}_j))) \longrightarrow K_*(\mathrm{Con}(C^\infty(\partial_j M) \to \Sigma_{M_j})) \xrightarrow{\partial'} K_{*+1}(S\Sigma_0) \qquad (26)$$

of the restriction of symbols to M_j and the boundary map ∂', where $\Sigma_{M_j} \equiv \Sigma(T^*M_j, \Omega)$ denotes the algebra of symbols of ψDO on Ω with parameters in T^*M_j and $\partial_j M \subset \partial M$ is the closure of $\pi^{-1}\mathcal{M}_j^\circ$.[5]

[5] One can readily see that $\partial_j M$ is a manifold fibered over M_j with fiber isomorphic to the blow-up of Ω.

3. The asymptotic homomorphism. Recall that $\Psi(T^*M_j, K_\Omega)$ is the algebra of jth symbols on M_j. To compute the boundary map ∂' in (26), we replace $\Psi(T^*M_j, K_\Omega)$ (preserving its K-group) by an algebra of pseudodifferential operators with parameters, for which the boundary map is simpler.

To this end, consider the map (see (13))

$$T_h : \Psi(T^*M_j \times \mathbb{R}, \Omega) \longrightarrow \Psi(T^*M_j, K_\Omega), \quad h \in (0, 1],$$

$$(T_h u)(\xi) = u\left(\overset{2}{r}\,\xi, ih\,\overset{1}{r\frac{\partial}{\partial r}} + ih(n+1)/2\right), \quad (\xi, p) \in T^*M_j \times \mathbb{R}, \tag{27}$$

where $\Psi(T^*M_j \times \mathbb{R}, \Omega)$ is the algebra of smooth families of ψDO on the fibers Ω with parameters in $T^*M_j \times \mathbb{R}$. As $h \to 0$, we have

$$T_h(ab) = T_h(a)T_h(b) + o(1), \quad (T_h(a))^* = T_h(a^*) + o(1), \tag{28}$$

where $a, b \in \Psi(T^*M_j \times \mathbb{R}, \Omega)$ are arbitrary and $o(1)$ is understood in the uniform norm.

Note that this semiclassical quantization is a special case of the so-called asymptotic homomorphisms, playing an important role in C^*-algebra theory [22–24]. In particular, Eqs. (28) imply that T_h induces a K-group homomorphism

$$T : K_*(\Psi(T^*M_j \times \mathbb{R}, \Omega)) \to K_*(\Psi(T^*M_j, K_\Omega)).$$

Consider the commutative diagram

$$\begin{array}{ccccccc}
0 \to & J(T^*M_j \times \mathbb{R}, \Omega) & \to & \Psi(T^*M_j \times \mathbb{R}, \Omega) & \to & \Sigma_{M_j} & \to 0 \\
& \downarrow T_h & & \downarrow T_h & & \downarrow t_h & \\
0 \to & \Sigma_0 & \to & \Psi(T^*M_j, K_\Omega) & \to & \Sigma_{M_j} & \to 0,
\end{array} \tag{29}$$

where t_h is the induced map on the symbols. (Actually, it is an algebra isomorphism.)

The algebra $C^\infty(\partial_j M)$ is a subalgebra in $\Psi(T^*M_j \times \mathbb{R}, \Omega), \Sigma_{M_j}$ and $\Psi(T^*M_j, K_\Omega)$. The diagram of the mapping cones of these embeddings gives the square

$$\begin{array}{ccc}
K_*(\mathrm{Con}(C^\infty(\partial_j M) \to \Sigma_{M_j})) & \xrightarrow{\partial''} & K_*(C_0(T^*M_j \times \mathbb{R})) \\
\| & & \downarrow T \\
K_*(\mathrm{Con}(C^\infty(\partial_j M) \to \Sigma_{M_j})) & \xrightarrow{\partial'} & K_{*+1}(S\Sigma_0).
\end{array} \tag{30}$$

(The horizontal maps are just the boundary maps in the corresponding sequences.) Here we have used the fact that

$$K_*(J(T^*M_j \times \mathbb{R}, \Omega)) \simeq K_*(C_0(T^*M_j \times \mathbb{R})).$$

This isomorphism is induced by the embedding $J(T^*M_j \times \mathbb{R}, \Omega) \subset C_0(T^*M_j \times \mathbb{R}, \mathcal{K})$ of a local C^*-algebra into its closure.

The commutativity in (30) follows, since the K-theory boundary map is natural with respect to asymptotic homomorphisms (e.g., see [25]).

4. The map $T : K_*(C_0(T^*M_j \times \mathbb{R})) \longrightarrow K_*(\Sigma_0)$ is an isomorphism. First, we compute $K_*(S\Sigma_0)$. To this end, consider the short exact sequence

$$0 \to \ker \sigma_c \longrightarrow \Sigma_0 \xrightarrow{\sigma_c} J(M_j \times \mathbb{R}, \Omega) \to 0 \tag{31}$$

of local C^*-algebras. (Here σ_c is the conormal symbol map, see Def. 2.7.) The kernel $\ker \sigma_c$ is formed by families of compact operators.[6] Since the embeddings

$$\ker \sigma_c \subset C(S^*M_j, \mathcal{K}(K_\Omega)) \quad \text{and} \quad J(M_j \times \mathbb{R}, \Omega) \subset C_0(M_j \times \mathbb{R}, \mathcal{K}(\Omega))$$

induce isomorphisms in K-theory, the K-theory long exact sequence induced by (31) can be written as the top row of the diagram

$$\begin{array}{ccccccc} \to & K^*(S^*M_j) & \to & K_*(\Sigma_0) & \to & K_c^*(M_j \times \mathbb{R}) & \to \\ & \| & & \downarrow L & & \| & \\ \to & K^*(S^*M_j) & \to & K_c^{*+1}(T^*M_j) & \to & K_c^*(M_j \times \mathbb{R}) & \to, \end{array} \tag{32}$$

where the bottom row is the sequence of topological K-groups of the pair $S^*M_j \subset B^*M_j$ of the unit sphere and ball bundles in T^*M_j and the map L is the difference construction for ψDO with operator-valued symbols in the sense of Luke (see [18,26]). Recall that this map is defined as follows: given $\sigma_j \in \Sigma_0^+$, we treat it as an operator-valued function on S^*M_j of compact variation in the fibers of $S^*_xM_j$ (see Proposition 2.2); if this function is invertible, then L takes $[\sigma_j] \in K_1(\Sigma_0)$ to the index

$$L[\sigma_j] := \operatorname{ind} \widetilde{\sigma_j} \in K_c(T^*M_j) \tag{33}$$

of an extension $\widetilde{\sigma_j}$ of σ_j to the unit ball bundle in T^*M_j preserving the compact variation property in the fibers. (The extension is Fredholm in B^*M_j and invertible on S^*M_j; thus its index is an element of the above mentioned K-group with compact supports.) The index (33) is independent of the choice of the extension. For even K-groups, the map is defined in a similar way.

Lemma 7.1. *The diagram (32) commutes, and hence L is an isomorphism.*

Proof. 1. The commutativity in

$$\begin{array}{ccc} K^*(S^*M_j) & \longrightarrow & K_*(\Sigma_0) \\ \| & & \downarrow L \\ K^*(S^*M_j) & \longrightarrow & K^{*+1}(T^*M_j) \end{array}$$

follows from the fact that for finite-dimensional symbols L coincides with the Atiyah–Singer difference construction.

2. Consider the square

$$\begin{array}{ccc} K_*(\Sigma_0) & \xrightarrow{\sigma_c} & K_c^*(M_j \times \mathbb{R}) \\ \downarrow L & & \| \\ K_c^{*+1}(T^*M_j) & \xrightarrow{j^*} & K^{*+1}(M_j), \end{array}$$

[6] Just as in the theory of operators on manifolds with isolated conical singularities, a family in Σ_0 is compact if and only if its conormal symbol is zero.

where $j : M_j \to T^*M_j$ is the embedding of the zero section. Its commutativity follows from the index formula (e.g., see [27])

$$\beta \operatorname{ind} D_y = \operatorname{ind} \sigma_c(D_y) \in K^1_c(Y \times \mathbb{R}) \tag{34}$$

for an elliptic family D_y, $y \in Y$ with unit interior symbol on the infinite cone. Here Y is a compact parameter space, and β is the periodicity isomorphism $K(Y) \simeq K^1_c(Y \times \mathbb{R})$.

Indeed, given $a \in K_1(\Sigma_0)$, the elements $j^*L(a)$ and $\sigma_c(a)$ are, respectively, the left- and the right- hand side in (34). The case of elements in the K_0-group can be verified if we first consider the suspension and then apply (34).

3. Now consider the square

$$\begin{array}{ccc} K_c^{*+1}(M_j \times \mathbb{R}) & \stackrel{\partial}{\longrightarrow} & K^*(S^*M_j) \\ \| & & \| \\ K^*(M_j) & \stackrel{p^*}{\longrightarrow} & K^*(S^*M_j) \end{array},$$

where $p : S^*M_j \to M_j$ is the natural projection. Its commutativity follows from the index formula. More precisely, let us represent $a \in K^1_c(M_j \times \mathbb{R})$ by an invertible family of conormal symbols with unit principal symbol. On the one hand, going through the left bottom corner of the square, we obtain $p^* \operatorname{ind} a \in K^0(S^*M_j)$. On the other hand, going through the upper right corner, we obtain

$$\partial a = p^* \operatorname{ind} \widehat{a},$$

where $\widehat{a}$ denotes the operator family on K_Ω with unit interior symbol and with conormal symbol a. Note that the latter equality follows from the fact that ∂ takes an invertible symbol to the index of the corresponding operator. Applying (34), we obtain the desired relation $p^* \operatorname{ind} a = p^* \operatorname{ind} \widehat{a} \in K^0(S^*M_j)$. For the case of elements in K_0 one should first use suspension. $\square$

Lemma 7.2. *The map*

$$T : K_*(C_0(T^*M_j \times \mathbb{R})) \longrightarrow K_*(\Sigma_0)$$

is the inverse of the isomorphism L in (32).

Proof. To be definite, we consider the map

$$T : K_1(C_0(T^*M_j \times \mathbb{R})) \longrightarrow K_1(\Sigma_0).$$

Let us prove that LT is the identity map.

1. Given a symbol $u(\xi, p) \in (1 + J(T^*M_j \times \mathbb{R}, \Omega))$ invertible for $(\xi, p) \in T^*M_j \times \mathbb{R}$ and equal to the identity in the complement of a compact set, we obtain

$$LT[u] = \operatorname{ind} u \left(\overset{2}{r}\, \xi, ih\, \overset{1}{r \frac{\partial}{\partial r}} + ih(n+1)/2 \right) \in K^0_c(T^*M_j)$$

(for h small), where $[u] \in K_c^1(T^*M_j \times \mathbb{R})$. The index is well defined, since $(T_h u)(\xi)$ has compact variation on the fibers in $T^*M_j \setminus \mathbf{0}$ and is invertible whenever $\xi \neq 0$ (see Proposition 2.2 and formula (14), respectively).

2. Let $\overline{T_h u}$ be the family of Fredholm operators parametrized by $T^*M_j \setminus \mathbf{0}$ and equal to $T_h u$ for $|\xi| < 1$ and to

$$u\left((\overset{2}{r} + |\xi| - 1)\xi, ih\, \overset{1}{r} \frac{\partial}{\partial r} + ih(n+1)/2 \right)$$

otherwise.

For small h, this family is invertible for all ξ. (This follows from the boundedness of the support of $1-u$ and since (14) is uniform for const $> \lambda \geq 0$ if in (13) we replace $r\xi$ by $(r+\lambda)\xi$). We obtain, by construction,

$$\operatorname{ind} T_h u = \operatorname{ind} \overline{T_h u}.$$

However, $\overline{T_h u}$ is the family of identity operators for large ξ. Hence its index can be calculated by (34) and is equal to the index of the family of conormal symbols $u(\xi, p)$ (modulo the Bott periodicity isomorphism); i.e., it actually gives the original element

$$LT[u] = [u].$$

$\square$

5. Comparison of boundary maps. Let us compare the obtained expressions for the boundary maps in K-homology and K-theory (see (25) and (26), (30)). We omit the restriction maps from Subsecs. 1 and 2. Consider the diagram (here and below, $I = (0,1)$)

$$\begin{array}{ccc}
K_*(\Xi \times I) & \xrightarrow{\pi'_*} & K_*(M_j^\circ \times I) \\
\varphi \uparrow & & \uparrow \varphi \\
K_*(\operatorname{Con}(C^\infty(\partial_j M) \to \Sigma_{M_j})) & \xrightarrow{\partial''} & K_c^*(T^*M_j \times \mathbb{R}) \\
\| & & \uparrow L \\
K_*(\operatorname{Con}(C^\infty(\partial_j M) \to \Sigma_{M_j})) & \xrightarrow{\partial'} & K_*(\Sigma_0),
\end{array} \tag{35}$$

containing the boundary maps in the middle and bottom rows. All maps in the diagram have already been defined except for φ in the left column. To define it, we note that $K_*(\operatorname{Con}(C^\infty(\partial_j M) \to \Sigma_{M_j}))$ classifies elliptic families $\sigma(x, \xi, p)$ on Ω with parameters $(x, \xi, p) \in T^*M_j \times \mathbb{R}$. Such a family defines an operator on the product $\Xi \times I$ by the formula

$$\sigma\left(x, -i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial t}\right).$$

This operator gives an element in $K_*(\Xi \times I)$ provided that $\sigma(x, \xi, p)$ is elliptic.

The bottom square in (35) is isomorphic to (30) by Lemma 7.2. Hence it commutes. We claim that the top square is also commutative. Indeed, consider any element $z \in K_*(\operatorname{Con}(C^\infty(\partial_j M) \to \Sigma_{M_j}))$ defined by an elliptic family $\sigma(x, \xi, p)$. Ellipticity implies the

Fredholm property, and ∂'' takes z simply to the index of the family with parameters in $T^*M_j \times \mathbb{R}$. On the other hand, $\pi'_* z$ corresponds to the elliptic operator

$$\sigma\left(x, -i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial t}\right)$$

in $C_0(M_j^\circ \times I)$-modules. The last two elements actually coincide. This is a consequence of the following general result.

Theorem 7.1. *Let $p(x,\xi)$ be an operator-valued symbol elliptic in the sense of Luke* [18] *on a compact manifold X with corners. Then*

$$\left[p\left(x, -i\frac{\partial}{\partial x}\right)\right] = \varphi\left(\operatorname{ind} p(x,\xi)\right) \in K_*(X^\circ), \tag{36}$$

where X° is the interior of the manifold, square brackets denote an element in K-homology, and $\varphi : K_c^*(T^*X) \to K_*(X^\circ)$ is the Poincaré isomorphism on manifolds with corners (e.g., see [28]*).*

The proof of this theorem is given in the supplement.

Thus Theorem 7.1 proves that (35) commutes. Hence we also have the commutativity of the squares in (24) containing the boundary maps.

This concludes the proof of Theorem 4.2.

8 Applications

A topological obstruction to Fredholm property. Let $\mathcal{M} \supset X$ be a stratified pair. It is of interest to find conditions under which an operator elliptic in $\mathcal{M} \setminus X$ can be transformed into an operator elliptic in $\mathcal{M}$ without changing the components of the principal symbol on the set $\mathcal{M}\setminus X$. This question is similar to the Atiyah–Bott problem of determining the topological conditions on the symbol on a smooth manifold with boundary under which there exists a Fredholm boundary condition for the corresponding operator.

We shall answer a similar question for the elements of Ell-groups in the case of an arbitrary stratification. To this end, consider the diagram

$$\begin{array}{ccccc}
\mathrm{Ell}(\mathcal{M}) & \longrightarrow & \mathrm{Ell}(\mathcal{M}, X) & & \\
\simeq\big\downarrow\varphi & & \varphi\big\downarrow\simeq & & \\
K_0(\mathcal{M}) & \longrightarrow & K_0(\mathcal{M}\setminus X) & \xrightarrow{\partial} & K_1(X).
\end{array}$$

It obviously commutes, since we deal with forgetful maps. Therefore, the nonvanishing of $\partial\varphi(a)$ is a necessary and sufficient condition for the existence of a lifting of $a \in \mathrm{Ell}(\mathcal{M}, X)$ to $\mathrm{Ell}(\mathcal{M})$.

The boundary map in K-homology plays a similar role in other problems (see Baum–Douglas [29], Roe [30], and Monthubert [31]).

Note that the equation $\partial\varphi(a) = 0$ is a condition on the interior symbol alone (a finite-dimensional condition) if $X = \mathcal{M}_{k-1}$ is the set of all singularities of $\mathcal{M} = \mathcal{M}_k$.

Cobordism invariance of the index. Let us give a generalization of the usual cobordism invariance of the index of Dirac operators. Suppose that X is a smooth stratum. Then we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Ell}_1(\mathcal{M}, X) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Ell}(X) \arrow[d, "\simeq"] & \\
K_1(\mathcal{M} \setminus X) \arrow[r] & K_0(X) \arrow[r] & K_0(\mathcal{M}).
\end{tikzcd}$$

Since the map $K_0(X) \to K_0(\mathcal{M})$ preserves the index (in $\mathbb{Z}$), we see that the index of D on X is zero provided that $[D] \in \mathrm{Ell}(X)$ can be pulled back to $\mathrm{Ell}_1(\mathcal{M}, X)$.

Remark 8.1. For nonsmooth X, the construction of this commutative diagram is an open problem. (Even the exact sequence in Ell-theory is not known.)

9 Supplement. Proof of Theorem 7.1

Note that Theorem 7.1 is a strengthening of the Luke theorem [18] for the index of operators with operator-valued symbols. (The index theorem is obtained from Eq. (36) if $\partial X = \emptyset$ and we consider only the indices of the corresponding K-homology elements.)

For the case of symbols homogeneous for large $|\xi|$, the proof carries over from [18] word for word. The general case (nonhomogeneous symbols) is reduced to the case of homogeneous symbols by the method suggested in [26], where it was adapted to the index computation. The reduction used in the present paper is based on the following standard fact of Kasparov's KK-theory.

Proposition 9.1. *Let P_t be a $*$-strongly continuous homotopy of bounded operators such that*

$$f[P_tP_t^* - 1], \quad f[P_t^*P_t - 1], \quad [f, P_t], \quad [f, P_t^*] \qquad \forall f \in C_0(X^\circ)$$

are norm continuous families of compact operators. Then the corresponding element in K-homology does not depend on the parameter:

$$[P_0] = [P_1] \in K_*(X^\circ).$$

Proof. It follows from the conditions that the family $\{P_t\}$ defines an operator in the $C_0(X^\circ) - C([0,1])$ bimodule $C_0(X^\circ \times [0,1], L^2(H))$. This operator defines a homotopy (in the sense of KK-theory) between P_0 and P_1 (e.g., see [32]). Hence the invariance of the K-homology element $[P_t]$ on t follows from the equivalence of the definitions of K-homology in terms of homotopies and in terms of operator homotopies. $\square$

Let us apply this proposition. Without loss of generality, we can assume that the symbol $p(x, \xi)$ is smooth up to the zero section in T^*X and normalized: $p^*(x,\xi)p(x,\xi) = 1$ for large $|\xi|$. Let $\psi(t)$, $t \geq 0$, be a smooth positive function such that

$$\psi(t) = \begin{cases} 1 & \text{for } t < 1, \\ 1/t & \text{for } t > 2. \end{cases} \tag{37}$$

Consider the family of symbols

$$p_\varepsilon(x,\xi) = p(x, \xi\psi(\varepsilon|\xi|)). \tag{38}$$

The following properties are obtained by a straightforward computation (cf. [33, Theorem 19.2.3]):

1. $p_0 = p$. For $\varepsilon > 0$, the symbol p_ε is homogeneous for large $|\xi|$, and for ε small it is elliptic.

2. p_ε and p_ε^* are uniformly bounded in the class of symbols of compact variation in the fibers of T^*X for $\varepsilon \in [0,1]$.

3. For small $\varepsilon > 0$, $p_\varepsilon p_\varepsilon^* - 1$ and $p_\varepsilon^* p_\varepsilon - 1$ are compactly supported and compact valued symbols independent of ε.

Consider the operator

$$P_\varepsilon = p_\varepsilon\left(x, -i\frac{\partial}{\partial x}\right). \tag{39}$$

(We fix an atlas of charts and a subordinate partition of unity independent of ε on X.) We claim that (for small $\varepsilon \geq 0$)

(a) $\operatorname{ind} p_\varepsilon \in K_c(T^*X)$ is independent of ε.

(b) P_ε satisfies the conditions of Proposition 9.1.

This will imply Theorem 7.1, since, on the one hand, p_ε is homogeneous at infinity for small $\varepsilon > 0$, and hence $[P_\varepsilon] = \varphi(\operatorname{ind} p_\varepsilon)$ (by the first part of the proof) and on the other hand, the passage to the limit as $\varepsilon \to 0$ is possible by Proposition 9.1. Thus it remains to prove (a) and (b).

Claim (a) follows from the homotopy invariance of the index, since p_ε changes under variations of ε only in the complement of a large ball $\{|\xi| > R\}$, where $R \simeq 1/\varepsilon$, and is invertible on this complement.

Claim (b) is proved as follows.

1) The families P_ε and P_ε^* are strongly continuous, since they are uniformly bounded and each summand in their definitions in coordinate patches on X is strongly continuous on the set of functions whose Fourier transform has compact support.

2) Given $f \in C_0(X^\circ)$, the operators $f(P_\varepsilon P_\varepsilon^* - 1)$ and $f(P_\varepsilon^* P_\varepsilon - 1)$ are compact and continuously depend on ε. Indeed, compactness is obvious, and the continuity follows from the fact that their complete symbols and their derivatives in local coordinates are uniformly continuous in ε on compact subsets in ξ, are uniformly bounded and decay to zero as $\xi \to \infty$, and hence, are uniformly continuous in ε for all ξ.

The compactness and continuity of the commutators $[f, P_\varepsilon]$ and $[f, P_\varepsilon^*]$ can be proved along the same lines.

The proof of Theorem 7.1 is complete.

References

[1] M. F. Atiyah. Global theory of elliptic operators. In *Proc. of the Int. Symposium on Functional Analysis*, pages 21–30, Tokyo, 1969. University of Tokyo Press.

[2] G. G. Kasparov. The generalized index of elliptic operators. *Funct. Anal. Appl.*, 7:238–240, 1973.

[3] L. Brown, R. Douglas, and P. Fillmore. Extensions of C^*-algebras and K-homology. *Ann. Math. II*, 105:265–324, 1977.

[4] I. M. Singer. Future extensions of index theory and elliptic operators. In *Prospects in Mathematics*, Ann. of Math. Studies, pages 171–185, Princeton, 1971.

[5] A. Savin and B. Sternin. Index defects in the theory of nonlocal boundary value problems and the η-invariant. *Sbornik:Mathematics*, 195(9), 2004. arXiv: math/0108107.

[6] A. Savin. Elliptic Operators on Singular Manifolds and K-homology. *K-theory*, 34(1):71–98, 2005.

[7] V. Nazaikinskii, A. Savin, B.-W. Schulze, and B. Sternin. *On the Homotopy Classification of Elliptic Operators on Manifolds with Edges.* Univ. Potsdam, Institut für Mathematik, Potsdam, 2004. Preprint N 2004/16, arXiv:math.OA/0503694.

[8] C. Debord and J.-M. Lescure. K-duality for pseudomanifolds with isolated singularities. *J. Funct. Anal.*, 219(1):109–133, 2005.

[9] R. Melrose and F. Rochon. Index in K-theory for families of fibred cusp operators. arxiv: math.DG/0507590, 2005.

[10] P. Baum and A. Connes. Geometric K-theory for Lie groups and foliations. *Enseign. Math. (2)*, 46(1–2):3–42, 2000.

[11] R. Lauter and V. Nistor. Analysis of geometric operators on open manifolds: a groupoid approach. In *Quantization of singular symplectic quotients*, volume 198 of *Progr. Math.*, pages 181–229. Birkhäuser, Basel, 2001.

[12] V. Nistor. Pseudodifferential operators on non-compact manifolds and analysis on polyhedral domains. In *Spectral geometry of manifolds with boundary and decomposition of manifolds*, volume 366 of *Contemp. Math.*, pages 307–328. Amer. Math. Soc., Providence, RI, 2005.

[13] R. Lauter, B. Monthubert, and V. Nistor. Pseudodifferential analysis on continuous family groupoids. *Doc. Math.*, 5:625–655 (electronic), 2000.

[14] J.-L. Tu. La conjecture de Baum-Connes pour les feuilletages moyennables. *K-Theory*, 17(3):215–264, 1999.

[15] B. A. Plamenevskij and V. N. Senichkin. On a class of pseudodifferential operators on $\mathbf{R}^m$ and on stratified manifolds. *Sb. Math.*, 191(5–6):725–757, 2000.

[16] B. A. Plamenevsky and V. N. Senichkin. Representations of C^*-algebras of pseudodifferential operators on piecewise-smooth manifolds. *Algebra i Analiz*, 13(6):124–174, 2001.

[17] Calvo D., Martin C.-I., and Schulze B.-W. Symbolic structures on corner manifolds. In *RIMS Conf. dedicated to L. Boutet de Monvel on Microlocal Analysis and Asymptotic Analysis, Kyoto, August 2004*, pages 22–35, Tokyo, 2005. Keio University.

[18] G. Luke. Pseudodifferential operators on Hilbert bundles. *J. Diff. Equations*, 12:566–589, 1972.

[19] V. Nazaikinskii, A. Savin, and B. Sternin. Pseudodifferential operators on stratified manifolds. Accepted for publication in *Differents. Uravn.,* arXiv:math.AP/0512025, 2005.

[20] N. Higson and J. Roe. *Analytic K-homology*. Oxford University Press, Oxford, 2000.

[21] M. Atiyah, V. Patodi, and I. Singer. Spectral asymmetry and Riemannian geometry III. *Math. Proc. Cambridge Philos. Soc.*, 79:71–99, 1976.

[22] A. Connes and N. Higson. Déformations, morphismes asymptotiques et K-théorie bivariante. *C. R. Acad. Sci.-Paris, Série 1*, 311(2):101–106, 1990.

[23] N. Higson. On the K-theory proof of the index theorem. In *Index Theory and Operator Algebras (Boulder, CO, 1991)*, volume 148 of *Contemp. Math.*, pages 67–86, Providence, RI, 1993. AMS.

[24] V. M. Manuilov. On asymptotic homomorphisms into Calkin algebras. *Funct. Anal. Appl.*, 35(2):148–150, 2001.

[25] A. Connes. *Noncommutative geometry*. Academic Press Inc., San Diego, CA, 1994.

[26] V. Nazaikinskii, A. Savin, B.-W. Schulze, and B. Sternin. On the index of differential operators on manifolds with edges. *Sbornik: Mathematics*, 196(9):1271–1305, 2005.

[27] V. Nazaikinskii, A. Savin, B.-W. Schulze, and B. Sternin. *Elliptic Theory on Manifolds with Nonisolated Singularities. III. The Spectral Flow of Families of Conormal Symbols*. Univ. Potsdam, Institut für Mathematik, Potsdam, 2002. Preprint N 2002/20.

[28] R. Melrose and P. Piazza. Analytic K-theory on manifolds with corners. *Adv. in Math.*, 92(1):1–26, 1992.

[29] P. Baum and R. G. Douglas. Index theory, bordism, and K-homology. In *Operator Algebras and K-Theory (San Francisco, Calif., 1981)*, volume 10 of *Contemp. Math.*, pages 1–31, Providence, RI, 1982. AMS.

[30] J. Roe. *Coarse cohomology and index theory on complete Riemannian manifolds*, volume 497 of *Mem. Amer. Math. Soc.* AMS, 1993.

[31] B. Monthubert. Groupoids of manifolds with corners and index theory. In *Groupoids in analysis, geometry, and physics (Boulder, CO, 1999)*, number 282 in Contemp. Math., pages 147–157, Providence, 2001. Amer. Math. Soc.

[32] B. Blackadar. *K-Theory for Operator Algebras.* Number 5 in Mathematical Sciences Research Institute Publications. Cambridge University Press, 1998. Second edition.

[33] L. Hörmander. *The Analysis of Linear Partial Differential Operators. III.* Springer–Verlag, Berlin Heidelberg New York Tokyo, 1985.